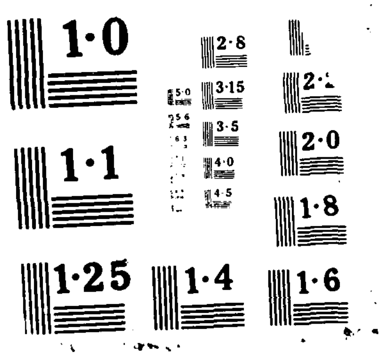
AD-A193 191 THE COMPUTATION OF STATIONARY DISTRIBUTIONS ON MARKOV CHAINS THROUGH PERT (U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS J J HUNTER MAR 88 TR-227 1/1

UNCLASSIFIED AFOSR-TR-88-0396 F49620-85-C-0144 F/G 12/3 NL


1·0
1·1
1·25
1·4
1·6
1·8
2·0
2·8
3·15
3·5
4·0

(2)

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

THE COMPUTATION OF STATIONARY DISTRIBUTIONS

ON MARKOV CHAINS THROUGH PERTURBATIONS

by

Jeffrey J. Hunter

Technical Report No. 227

March 1988

145. C. Kallianpur and V. Perez-Abreu, Stochastic evolution equations with values on the dual of a countably Hilbert nuclear space. July 86. Appl. Math. Optimization, to appear.

146. B.G. Nguyen, Fourier transform of the percolation free energy. July 86. Probab. Theor. Rel. Fields, to appear.

147. A.M. Hasofer, Distribution of the maximum of a Gaussian process by Monte Carlo. July 86. J. Sound Vibration, 112, 1987, 283-293.

148. T. Norberg, On the existence of probability measures on continuous semi-lattices. Aug. 86.

149. G. Samorodnitsky, Continuity of Gaussian processes, Aug. 86. Ann. Probability, to appear.

150. T. Hsing, J. Husler, and M.R. Leadbetter, Limits for exceedance point processes, Sept. 86. Prob. Theory and Related Fields, to appear.

151. S. Cambanis, Random filters which preserve the stability of random inputs, Sept. 86. Adv. Appl. Probability, 1987, to appear.

152. O. Kallenberg, On the theory of conditioning in point processes, Sept. 86. Proceedings for the First World Congress of the Bernoulli Society, Tashkent, 1986.

153. G. Samorodnitsky, Local moduli of continuity for some classes of Gaussian processes, Sept. 86.

154. V. Mandrekar, On the validity of Beurling theorems in polydiscs, Sept. 86.

155. R.F. Serfozo, Extreme values of queue lengths in M/G/1 and GI/M/1 systems, Sept. 86.

156. B.G. Nguyen, Gap exponents for percolation processes with triangle condition, Sept. 86. J. Statist. Physics, 49, 1987, 235-243.

157. G. Kallianpur and R. Wolpert, Weak convergence of stochastic neuronal models, Oct. 86. Stochastic Methods in Biology, M. Kimura et al., eds., Lecture Notes in Biomathematics, 70, Springer, 1987, 116-145.

158. G. Kallianpur, Stochastic differential equations in duals of nuclear spaces with some applications, Oct. 86. Inst. of Math. & Its Applications, 1986.

159. G. Kallianpur and R.L. Karandikar, The filtering problem for infinite dimensional stochastic processes, Jan. 87. Proc. Workshop on Stochastic Differential Systems, Stochastic Control Theory & Applications, Springer, to appear.

160. V. Perez-Abreu, Multiple stochastic integrals and nonlinear functionals of a nuclear space valued Wiener process, Oct. 86. Appl. Math. Optimization, 16, 1987, 227-245.

161. R.L. Karandikar, On the Feynman-Kac formula and its applications to filtering theory, Oct. 86. Appl. Math. Optimization, to appear.

162. R.L. Taylor and T.-C. Hu, Strong laws of large numbers for arrays of rowwise independent random elements, Nov. 86. Intern. J. Math. & Math. Sci., to appear.

163. M. O'Sullivan and T.R. Fleming, Statistics for the two-sample survival analysis problem based on product limit estimators of the survival functions, Nov. 86.

164. F. Avram, On bilinear forms in Gaussian random variables, Toeplitz matrices and Parseval's relation, Nov. 86.

165. D.B.H. Cline, Joint stable attraction of two sums of products, Nov. 86. J. Multivariate Anal., to appear.

166. R.J. Wilson, Model fields in crossing theory-a weak convergence perspective, Nov. 86.

167. D.B.H. Cline, Consistency for least squares regression estimators with infinite variance data, Dec. 86.

168. L.L. Campbell, Phase distribution in a digital frequency modulation receiver, Nov. 86.

169. B.G. Nguyen, Typical cluster size for 2-dim percolation processes, Dec. 86. J. Statist. Physics, to appear.

170. H. Oodaira, Freidlein-Wentzell type estimates for a class of self-similar processes represented by multiple Wiener integrals, Dec. 86.

171. J. Nolan, Local properties of index-β stable fields, Dec. 86. Ann. Probability, to appear.

172. R. Menich and R.F. Serfozo, Optimality of shortest queue routing for dependent service stations, Dec. 86.

173. F. Avram and M.S. Taqqu, Probability bounds for M-Skorohod oscillations, Dec. 86.

174. F. Moricz and R.L. Taylor, Strong laws of large numbers for arrays of orthogonal random variables, Dec. 86.

175. G. Kallianpur and V. Perez-Abreu, Stochastic evolution driven by nuclear space valued martingales, Apr. 87.

176. E. Merzbach, Point processes in the plane, Feb. 87.

177. Y. Kasahara, M. Maejima and W. Vervaat, Log fractional stable processes, March 87.

178. G. Kallianpur, A.G. Miamee and H. Niemi, On the prediction theory of two parameter stationary random fields, March 87.

179. R. Brigola, Remark on the multiple Wiener integral, Mar. 87.

180. R. Brigola, Stochastic filtering solutions for ill-posed linear problems and their extension to measurable transformations, Mar. 87.

181. G. Samorodnitsky, Maxima of symmetric stable processes, Mar. 87.

182. H.L. Hurd, Representation of harmonizable periodically correlated processes and their covariance, Apr. 87.

183. H.L. Hurd, Nonparametric time series analysis for periodically correlated processes, Apr. 87.

184. T. Mori and H. Oodaira, Freidlin-Wentzell estimates and the law of the iterated logarithm for a class of stochastic processes related to symmetric statistics, May 87.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: NA	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release; Distribution Unlimited
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 227	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 88-0396
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina; 6b. OFFICE SYMBOL (If applicable):	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State and ZIP Code): Statistics Department, CB #3260, Phillips Hall, Chapel Hill, NC 27599-3260	7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144
8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, DC	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
6.1102F	2304	A5	leave blnk

11. TITLE (Include Security Classification): The computation of stationary distributions on Markov chains through perturbations

12. PERSONAL AUTHOR(S): Hunter, J.J.

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
preprint	FROM 9/87 TO 8/88	Mar. 1988	18

16. SUPPLEMENTARY NOTATION: N/A

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Key words and phrases: N/A
XXXXXXXXXXXXXX			

19. ABSTRACT

An algorithmic procedure, for the determination of the stationary distribution of a finite, m-state, irreducible Markov chain, that does not require the use of methods for solving systems of linear equations, is presented. The technique is based upon a succession of m, rank one, perturbations of the trivial doubly stochastic matrix whose known steady state vector is updated at each stage to yield the required stationary probability vector.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Major Brian W. Woodruff	22b. TELEPHONE NUMBER (Include Area Code): (202) 767-5026; 22c. OFFICE SYMBOL: AFOSR/NM

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

THE COMPUTATION OF STATIONARY DISTRIBUTIONS

OF MARKOV CHAINS THROUGH PERTURBATIONS

by

Jeffrey J. Hunter
Department of Mathematics and Statistics
University of Auckland

and

Center for Stochastic Processes
Department of Statistics
University of North Carolina

ABSTRACT

An algorithmic procedure, for the determination of the stationary distribution of a finite, m-state, irreducible Markov chain, that does not require the use of methods for solving systems of linear equations, is presented. The technique is based upon a succession of m, rank one, perturbations of the trivial doubly stochastic matrix whose known steady state vector is updated at each stage to yield the required stationary probability vector.

This research supported in part by the Air Force Office of Scientific Research Contract No. F49620 85 C 0144.

1. Introduction

In recent years, widespread attention has been given to the computation of stationary distributions of Markov chains. A variety of methods have been suggested and implemented. Before considering an alternative way for finding such distributions, it is of interest to give a brief survey of the techniques that have been employed.

Paige, Styan and Wachter (1975) presented a comprehensive survey of eight different algorithms involving a variety of procedures including the use of generalized inverses, rank reduction, least squares and power methods. Their recommendation was a direct method that involved transforming the singular set of stationary equations into a non-singular system using a rank one modification followed by Gaussian elimination with row pivoting. A further study by Harrod and Plemmons (1984) provided another direct approach based upon the LU factorization using Gaussian elimination without pivoting.

Iterative techniques and approximation methods have been surveyed by Koury, McAllister and Stewart (1984). When the transition matrix is large and exhibits a nearly completely decomposable structure it is shown that a method of "aggregation" can be combined with point and block iterative techniques to produce methods which converge rapidly to the stationary probability vector.

Sheskin (1985) presented a partitioning algorithm that used a matrix reduction routine that partitions the transition matrix to create a sequence of smaller order transition matrices followed by a vector enlargement routine that enables the components of the steady state vector to be determined sequentially. A related procedure was developed by Grassmann, Taksar and Heyman (1985) using the theory of regenerative processes. They derived relationships between the steady state probabilities which are then used to develop a numerical algorithm to find these probabilities. Both of these latter two techniques appear to be, in effect, modifications of Gaussian elimination.

More recently, Meyer (1987), has utilised the concept of "stochastic complementation" whereby an irreducible chain is uncoupled into smaller irreducible chains whose stationary distributions can be coupled back together to produce the stationary distribution of the original chain.

In this paper an entirely new approach involving the analysis of perturbed Markov chains is considered. In Hunter (1986) techniques for updating the stationary distribution of a finite irreducible Markov chain, following a rank one perturbation of its transition matrix, were presented. In this current paper, these techniques are utilised, to construct a general procedure for determining the stationary distribution of any finite irreducible Markov chain. A significant feature of the proposed algorithm is that at no stage does a system of linear equations have to be solved and consequently there is no reliance upon computer subroutines for matrix inversion or, the more generally accepted method of solution, Gaussian elimination with or without pivoting.

The basic idea is very simple. Suppose the steady state probability vector $\underset{\sim}{\pi}'$ of an m-state irreducible Markov chain with given transition matrix P is required. Let P_o be the transition matrix of another irreducible, m-state, Markov chain with known stationary probability vector $\underset{\sim}{\pi}_o'$. By replacing, successively, the elements of each row of P_o with the corresponding row elements as specified by P and recomputing the stationary probability vector of the resultant perturbed transition matrix, the vector $\underset{\sim}{\pi}_o'$ can be transformed, in m stages, to $\underset{\sim}{\pi}'$, by a series of m updates.

As the irreducibility of a Markov chain is governed by the location of the positive entries in its transition matrix, to ensure the irreducibility of each perturbed Markov chain it is sufficient to commence with P_o containing positive elements placed at least in the same position as those in P.

Consider starting with the trivial doubly stochastic matrix P_o with each element having the value 1/m, so that $P_o = \underset{\sim}{e}\underset{\sim}{e}'/m$, where $\underset{\sim}{e}' = (1,1,\ldots,1)$ is a vector of ones. As can be easily shown, $\underset{\sim}{\pi}_o' = \underset{\sim}{e}'/m$.

For $i=1,2,\ldots,m$, let $\underset{\sim}{e}_i$ be the i^{th} elementary (column) vector with a one in the i^{th} position and zeroes elsewhere. Let $\underset{\sim}{p}_i' = \underset{\sim}{e}_i'P$ be the i^{th} row of P and let

$$P_i = P_{i-1} + \underset{\sim}{e}_i\underset{\sim}{b}_i', \tag{1.1}$$

where

$$\underset{\sim}{b}_i' = \underset{\sim}{p}_i' - \underset{\sim}{e}'/m.$$

Let $\underset{\sim}{\pi}_i'$ be the stationary probability vector associated with the Markov chain with transition matrix P_i, and, since $P_m = \sum_{i=1}^{m} \underset{\sim}{e}_i\underset{\sim}{p}_i' = P$, $\underset{\sim}{\pi}_m'$ is in fact the required vector $\underset{\sim}{\pi}'$.

2. General Theory

The construction of the algorithm is based upon the following results.

Theorem 2.1: Let P_i be the transition matrix of a finite irreducible Markov chain with stationary probability vector $\underset{\sim}{\pi}_i'$.

(a) $I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i'$ is non-singular if and only if $\underset{\sim}{u}_i'\underset{\sim}{e} \neq 0$ and $\underset{\sim}{\pi}_i'\underset{\sim}{t}_i \neq 0$.

(b) Under the conditions of (a),

$$\underset{\sim}{\pi}_i' = \underset{\sim}{\alpha}_i'/\underset{\sim}{\alpha}_i'\underset{\sim}{e}, \tag{2.1}$$

where

$$\underset{\sim}{\alpha}_i' = \underset{\sim}{u}_i'[I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i']^{-1}. \tag{2.2}$$

Proof: For (a) see Theorem 3.3 in Hunter (1982) and for (b) see Corollary 4.1.2 in Hunter (1982). □

Theorem 2.2 If X is non-singular and $\underset{\sim}{b}'X^{-1}\underset{\sim}{a} \neq -1$, then

$$(X + \underset{\sim}{a}\underset{\sim}{b}')^{-1} = X^{-1} - \frac{X^{-1}\underset{\sim}{a}\underset{\sim}{b}'X^{-1}}{1 + \underset{\sim}{b}'X^{-1}\underset{\sim}{a}} . \tag{2.3}$$

Proof. This is the Sherman-Morrison formula. See Golub and Van Loan (1983) p3.

□

Suppose that, following the i^{th} perturbation, the stationary probability vector $\underset{\sim}{\pi}_i'$ has been found for the Markov chain with transition matrix P_i, as given by (1.1), by using the procedure described by Theorem 2.1(b) for suitable choices of $\underset{\sim}{t}_i$ and $\underset{\sim}{u}_i$.

In Hunter (1986) it was shown that it is possible to find an expression for $\underset{\sim}{\pi}_{i+1}'$, associated with P_{i+1}, using the same procedure outlined in Theorem 2.1(b), by choosing the $\underset{\sim}{t}_{i+1}$ and $\underset{\sim}{u}_{i+1}$ in such a way that $[I - P_{i+1} + \underset{\sim}{t}_{i+1}\underset{\sim}{u}_{i+1}']^{-1}$ can be determined from the earlier deduced expression $[I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i']^{-1}$, without performing an additional matrix inversion.

For the particular situation under consideration, for i=0,1,...,m-1, if $\underset{\sim}{t}_{i+1} = \underset{\sim}{e}_{i+1}$ and $\underset{\sim}{u}_{i+1} = \underset{\sim}{u}_i + \underset{\sim}{b}_{i+1}$, where $\underset{\sim}{b}_i$ is given by (1.2), then, from (1.1),

$$\begin{aligned} I - P_{i+1} + \underset{\sim}{t}_{i+1}\underset{\sim}{u}_{i+1}' &= I - P_i + \underset{\sim}{e}_{i+1}\underset{\sim}{u}_i', \\ &= I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i' + (\underset{\sim}{t}_{i+1} - \underset{\sim}{t}_i)\underset{\sim}{u}_i'. \end{aligned} \tag{2.4}$$

Now if $[I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i']^{-1}$ exists, from the proof of Theorem 3.3 in Hunter (1982),

$$\underset{\sim}{u}_i'[I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i']^{-1} = \frac{\underset{\sim}{\pi}_i'}{\underset{\sim}{\pi}_i'\underset{\sim}{t}_i}. \tag{2.5}$$

Thus, using (2.3), (2.4) and (2.5), if $A_i \equiv [I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}_i']^{-1}$ exists,

$$A_{i+1} = A_i[I + (\underset{\sim}{t}_i - \underset{\sim}{t}_{i+1}) \frac{\underset{\sim}{\pi}_i'}{\underset{\sim}{\pi}_i'\underset{\sim}{t}_{i+1}}] \quad . \tag{2.6}$$

Equation (2.6) is ideally suited for recursive operations once an initial inverse $A_o = [I - P_o - \underset{\sim}{t}_o\underset{\sim}{u}_o']^{-1}$ has been determined. However, because of the form of P_o that has been selected, if $\underset{\sim}{t}_o = \underset{\sim}{e}$ and $\underset{\sim}{u}_o' = \underset{\sim}{e}'/m$, no matrix inverse has to be computed since, in this instance,

$$I - P_o + \underset{\sim}{t}_o\underset{\sim}{u}_o' = I - \underset{\sim}{e}\underset{\sim}{e}'/m + \underset{\sim}{e}\underset{\sim}{e}'/m = I.$$

Furthermore, using (2.2), $\underset{\sim}{\alpha}_o' = \underset{\sim}{u}_o' = \underset{\sim}{e}'/m$,

and, from (2.1), $\underset{\sim}{\pi}_o' = \underset{\sim}{e}'/m$.

The basic algorithmic procedure now follows:

Let $\underset{\sim}{t}_o = \underset{\sim}{e}$, $\underset{\sim}{u}_o' = \underset{\sim}{e}'/m$, $A_o = I$, $\underset{\sim}{\pi}_o' = \underset{\sim}{e}'/m$.

For i=1,2,...,m, let $\underset{\sim}{t}_i = \underset{\sim}{e}_i$, and $\underset{\sim}{u}_i' = \underset{\sim}{u}_{i-1}' + \underset{\sim}{p}_i' - \underset{\sim}{e}'/m$.

Compute
$$A_i = A_{i-1}[I + (\underset{\sim}{t}_{i-1} - \underset{\sim}{t}_i)\underset{\sim}{\pi}_{i-1}'/\underset{\sim}{\pi}_{i-1}'\underset{\sim}{t}_i]. \tag{2.7}$$

Compute
$$\underset{\sim}{\alpha}_i' = \underset{\sim}{u}_i'A_i. \tag{2.8}$$

Compute
$$\underset{\sim}{\pi}_i' = \underset{\sim}{\alpha}_i'/\underset{\sim}{\alpha}_i'\underset{\sim}{e}. \tag{2.9}$$

Then $\underset{\sim}{\pi}' = \underset{\sim}{\pi}_m'$ is the stationary probability vector of the Markov chain with transition matrix $P = [p_{ij}]$.

Since the elements of any stationary probability vector are always positive, $\underset{\sim}{\pi}_i'\underset{\sim}{t}_i$ and $\underset{\sim}{\pi}_{i-1}'\underset{\sim}{t}_i$ are both positive. Further, by induction, for i=0,1,...,m,

$$\underset{\sim}{u}_i'\underset{\sim}{e} = 1, \tag{2.10}$$

so that the conditions of Theorem 2.1 and 2.2 are satisfied.

3. Refinements to the algorithm

Although the procedure suggested in Section 2 will lead to the required stationary probability vector there are some modifications that, if employed, will lead to a more efficient procedure.

3.1 Modification to the $\underset{\sim}{\pi}_i'$ computation

The ultimate aim of the algorithm is to determine $\underset{\sim}{\pi}' = \underset{\sim}{\pi}_m'$. Unless the stationary distributions of the intermediate perturbed Markov chains are required, some simplification can be effected by observing that (2.7) requires $\underset{\sim}{\pi}_{i-1}'$ through its scaled version $\underset{\sim}{\pi}_{i-1}'/\underset{\sim}{\pi}_{i-1}'\underset{\sim}{t}_i$. The scaling suggested by (2.9) is not required until the final step when i=m.

Thus, for i=0,1,...,m-1, let

$$\underset{\sim}{\upsilon}_i' \equiv \underset{\sim}{\pi}_i'/\underset{\sim}{\pi}_i'\underset{\sim}{t}_{i+1} . \qquad (3.1)$$

Then, $\underset{\sim}{\upsilon}_o' = \underset{\sim}{e}'$ and for i=1,2,...,m-1,

$$\underset{\sim}{\upsilon}_i' = \underset{\sim}{u}_i'A_i/\underset{\sim}{u}_i'A_i\underset{\sim}{e}_{i+1} . \qquad (3.2)$$

If $\underset{\sim}{\pi}_i'$ is required then it can be recovered simply. For i=0,1,...,m-1,

$$\underset{\sim}{\pi}_i' = \underset{\sim}{\upsilon}_i'/\underset{\sim}{\upsilon}_i'\underset{\sim}{e} . \qquad (3.3)$$

At the final step, compute $\underset{\sim}{\pi}_m'$ using (2.8) and (2.9).

3.2 Modification to the A_i computation

With the notation introduced in Section 3.1, it is easily seen that the early terms in the $\{A_i\}$ sequence are given, after simplification, as

$$A_o = I,$$

$$A_1 = A_o + (\underset{\sim}{e} - \underset{\sim}{e}_1)\underset{\sim}{\upsilon}_o' . \qquad (3.4)$$

$$A_2 = A_1 + (\underset{\sim}{e}_1 - \underset{\sim}{e}_2)\underset{\sim}{\upsilon}_1' . \qquad (3.5)$$

$$A_3 = A_2 + [(1 - \upsilon_{13})\underset{\sim}{e}_1 + \upsilon_{13}\underset{\sim}{e}_2 - \underset{\sim}{e}_3]\underset{\sim}{\upsilon}_2' . \qquad (3.6)$$

$$A_4 = A_3 + [\{1 - \upsilon_{14} - (1 - \upsilon_{13})\upsilon_{24}\}\underset{\sim}{e}_1 + (\upsilon_{14} - \upsilon_{13}\upsilon_{24})\underset{\sim}{e}_2 + \upsilon_{24}\underset{\sim}{e}_3 - \underset{\sim}{e}_4]\underset{\sim}{\upsilon}_3', \tag{3.7}$$

where $\upsilon_{ij} \equiv \underset{\sim}{\upsilon}_i'\underset{\sim}{e}_j$, so that $\upsilon_{i,i+1} = 1$ for $i=1,2,\ldots,m-1$.

The above results provide motivation for the following theorem.

Theorem 3.1: For $n=0,1,\ldots,m-1$,

$$A_{n+1} = A_n + B_n, \tag{3.8}$$

where

$$B_n = A_n(\underset{\sim}{e}_n - \underset{\sim}{e}_{n+1})\underset{\sim}{\upsilon}_n' \equiv \underset{\sim}{b}_n\,\underset{\sim}{\upsilon}_n' \tag{3.9}$$

with $\underset{\sim}{e}_o \equiv \underset{\sim}{e}$, so that $\underset{\sim}{b}_o = \underset{\sim}{e} - \underset{\sim}{e}_1$ and for $n=1,2,\ldots,m-1$,

$$\underset{\sim}{b}_n = b_{1n}\underset{\sim}{e}_1 + \ldots + b_{nn}\underset{\sim}{e}_n - \underset{\sim}{e}_{n+1}\,. \tag{3.10}$$

Proof: The theorem is obviously true by inspection, from (3.4) to (3.7), for $n=0,1,2,3$. Assume that (3.9) and (3.10) hold for $n=0,1,\ldots,k$ so that

$$A_{k+1} = A_k + B_k = A_{k-1} + B_{k-1} + B_k,$$
$$= \ldots = I + \sum_{n=0}^{k} B_n. \tag{3.11}$$

Hence

$$B_{k+1} = A_{k+1}(\underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2})\underset{\sim}{\upsilon}_{k+1}',$$
$$= (I + B_o + \sum_{n=1}^{k} B_n)(\underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2})\underset{\sim}{\upsilon}_{k+1}',$$

implying that

$$\underset{\sim}{b}_{k+1} = [I + (\underset{\sim}{e} - \underset{\sim}{e}_1)\underset{\sim}{e}' + \sum_{n=1}^{k}(\sum_{m=1}^{n} b_{mn}\underset{\sim}{e}_m - \underset{\sim}{e}_{n+1})\underset{\sim}{\upsilon}_n'](\underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2}).$$

Since $\underset{\sim}{e}'(\underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2}) = 0$,

$$\underset{\sim}{b}_{k+1} = \underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2} + \sum_{n=1}^{k}(\sum_{m=1}^{n} b_{mn}\underset{\sim}{e}_m - \underset{\sim}{e}_{n+1})(\upsilon_{n,k+1} - \upsilon_{n,k+2}),$$
$$= \underset{\sim}{e}_{k+1} - \underset{\sim}{e}_{k+2} + \sum_{m=1}^{k}\{\sum_{n=m}^{k} b_{mn}(\upsilon_{n,k+1} - \upsilon_{n,k+2})\}\underset{\sim}{e}_m$$
$$- \sum_{m=2}^{k+1}(\upsilon_{m-1,k+1} - \upsilon_{m-1,k+2})\underset{\sim}{e}_m\,.$$

showing that $\underset{\sim}{b}_{k+1}$ is a linear combination of $\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_{k+2}$ with $\underset{\sim}{e}_{k+2}$ having coefficient -1. Thus (3.10) is true for n=k+1 and the theorem follows by induction.

Observe that

$$\underset{\sim}{b}_{k+1} = \sum_{m=1}^{k+1} b_{m,k+1}\underset{\sim}{e}_m - \underset{\sim}{e}_{k+2}, \tag{3.12}$$

where

$$b_{1,k+1} = \sum_{n=1}^{k} b_{1,n}(\upsilon_{n,k+1} - \upsilon_{n,k+2}) , \tag{3.13}$$

$$b_{k+1,k+1} = \upsilon_{k,k+2} \text{ , (since } \upsilon_{k,k+1}=1), \tag{3.14}$$

and for m=2,...,k,

$$b_{m,k+1} = \sum_{n=m-1}^{k} b_{m,n}(\upsilon_{n,k+1} - \upsilon_{n,k+1}) . \tag{3.15}$$ □

Note that

$$B_n = \underset{\sim}{b}_n \underset{\sim}{\upsilon}'_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\upsilon_{n1},\upsilon_{n2},\ldots,\upsilon_{nm}] ,$$

$$= \begin{bmatrix} b_{1n}\upsilon_{n1} & b_{1n}\upsilon_{n2} & \cdots & b_{1n}\upsilon_{nm} \\ b_{2n}\upsilon_{n1} & b_{2n}\upsilon_{n2} & \cdots & b_{2n}\upsilon_{nm} \\ \vdots & \vdots & \vdots & \vdots \\ b_{nn}\upsilon_{n1} & b_{nn}\upsilon_{n2} & \cdots & b_{nn}\upsilon_{nm} \\ -\upsilon_{n1} & -\upsilon_{n2} & \cdots & -\upsilon_{nm} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} . \tag{3.16}$$

Thus, in the matrix B_n, all the entries in rows numbered n+2,...,m are zero. Obviously, this has considerable significance in the calculation of the matrices A_i (i=2,...,m) as required in the algorithm.

The updating process, given by (2.7), can be replaced for i=2,...,m, by

$$A_i = A_{i-1} + B_{i-1}, \tag{3.17}$$

where

$$B_{i-1} = A_{i-1}(\underset{\sim}{e}_{i-1} - \underset{\sim}{e}_i)\underset{\sim}{\upsilon}'_{i-1}, \tag{3.18}$$

is such that only the first i rows of B_{i-1} are computed with the remaining entries set equal to zero.

Furthermore, from (3.16), some of the rows of B_n have a special form and do not require computation. In particular the (n+1)th row is simply $-\underset{\sim}{\upsilon}'_n$ while, from (3.14), the n^{th} row is $\upsilon_{n-1,n+1}$ times $\underset{\sim}{\upsilon}'_n$.

Note also that the (n+1)st column of B_n is $\underset{\sim}{b}_n$ since $\upsilon_{n,n+1}=1$.

There are also some other checks that can be applied.

<u>Theorem 3.2:</u> For i=1,2,...,m,

$$A_i\underset{\sim}{e}_i = \underset{\sim}{e}, \tag{3.19}$$

$$\frac{1}{m}\underset{\sim}{e}'A_i = \underset{\sim}{e}, \tag{3.20}$$

$$\underset{\sim}{e}'B_i = \underset{\sim}{0}'. \tag{3.21}$$

<u>Proof:</u> Since $A_i = [I - P_i + \underset{\sim}{t}_i\underset{\sim}{u}'_i]^{-1}$, equation (3.17) of Hunter (1982) implies that

$$A_i\underset{\sim}{t}_i = \frac{\underset{\sim}{e}}{\underset{\sim}{u}'_i\underset{\sim}{e}} = \underset{\sim}{e},$$

yielding (3.19) with $\underset{\sim}{t}_i = \underset{\sim}{e}_i$.

Equation (3.20) is obviously true when i=1 since

$$\begin{aligned}\underset{\sim}{e}'A_1 &= \underset{\sim}{e}'[I + (\underset{\sim}{e} - \underset{\sim}{e}_1)\underset{\sim}{e}'],\\ &= \underset{\sim}{e}' + (m-1)\underset{\sim}{e}',\\ &= m\,\underset{\sim}{e}'.\end{aligned}$$

Thus, by induction, if (3.20) is true for i=1,2,...,k, from (2.7) and (3.1),

$$\frac{1}{m}\underset{\sim}{e}'A_{k+1} = \frac{1}{m}\underset{\sim}{e}\, A_k[I + (\underset{\sim}{e}_k - \underset{\sim}{e}_{k+1})\underset{\sim}{\upsilon}_k'] ,$$
$$= \underset{\sim}{e}'[I + (\underset{\sim}{e}_k - \underset{\sim}{e}_{k+1})\underset{\sim}{\upsilon}_k'] ,$$
$$= \underset{\sim}{e}' + (\underset{\sim}{e}'\underset{\sim}{e}_k - \underset{\sim}{e}'\underset{\sim}{e}_{k+1})\underset{\sim}{\upsilon}_k' = \underset{\sim}{e}' .$$

Thus (3.20) and hence also (3.21) follow. □

A consequence of Theorem 3.2 is that the i^{th} column of A_i consists solely of unit elements while the sum of the elements of each column of B_i is zero.

3.3 Modification to the $\underset{\sim}{\alpha}_i$ computation

Although the $\{A_i\}$ (i=0,1,...,m) sequence plays an integral role in the procedure, the matrices A_i are required only to obtain the sequence of vectors $\underset{\sim}{\alpha}_i' = \underset{\sim}{u}_i'A_i$ and hence the vectors $\underset{\sim}{\upsilon}_i'$. Thus is is worth examining whether it is possible to dispense with explicit calculation of the A_i by deriving the $\{\underset{\sim}{\alpha}_i'\}$ (i=1,2,...,m) sequence recursively.

Theorem 3.3: For i=0,1,2,...,m-1,

$$\underset{\sim}{\alpha}_{i+1}' = \underset{\sim}{\upsilon}_i' - \underset{\sim}{e}' + \underset{\sim}{p}_{i+1}'A_{i+1} . \tag{3.22}$$

Proof: First observe that, from (2.8),

$$\underset{\sim}{\alpha}_1' = \underset{\sim}{u}_1'A_1 = \underset{\sim}{p}_1'A_1,$$

so that (3.22) holds for i=0 since $\underset{\sim}{\upsilon}_o' = \underset{\sim}{e}'$.

In general, for i=1,2,...,m-1, from (2.8),

$$\underset{\sim}{\alpha}_{i+1}' = \underset{\sim}{u}_{i+1}'A_{i+1},$$
$$= [\underset{\sim}{u}_i' - (\underset{\sim}{e}'/m) + \underset{\sim}{p}_{i+1}']A_{i+1} . \tag{3.23}$$

Now
$$\underset{\sim}{u}_i'A_{i+1} = \underset{\sim}{u}_i'[A_i + A_i(\underset{\sim}{e}_i - \underset{\sim}{e}_{i+1})\underset{\sim}{\upsilon}_i'],$$
$$= \underset{\sim}{\alpha}_i' + \underset{\sim}{\alpha}_i'(\underset{\sim}{e}_i - \underset{\sim}{e}_{i+1})\underset{\sim}{\upsilon}_i',$$
$$= \underset{\sim}{\alpha}_i' + (\alpha_{ii} - \alpha_{i,i+1})\underset{\sim}{\upsilon}_i' . \tag{3.24}$$

where $\alpha_{ij} = \underset{\sim}{\alpha}'_i \underset{\sim}{e}_j$. But, from (3.19) and (2.10),

$$\alpha_{ii} = \underset{\sim}{u}'_i A_i \underset{\sim}{e}_i = \underset{\sim}{u}'_i \underset{\sim}{e} = 1 \ , \tag{3.25}$$

and, since from (3.2) and (2.8), $\underset{\sim}{\upsilon}'_i = \underset{\sim}{\alpha}'_i / \alpha_{i,i+1}$, (3.24) becomes

$$\underset{\sim}{u}'_i A_{i+1} = \underset{\sim}{\upsilon}'_i . \tag{3.26}$$

Equation (3.22) now follows from (3.23) upon substitution of (3.26) and (3.20). □

Theorem 3.3 shows that in updating from $\underset{\sim}{\alpha}'_i$ to $\underset{\sim}{\alpha}'_{i+1}$ the term $\underset{\sim}{p}'_{i+1} A_{i+1}$ must be computed. The calculations of $\underset{\sim}{\alpha}'_1, \underset{\sim}{\alpha}'_2, \ldots, \underset{\sim}{\alpha}'_i$ require, successively, $\underset{\sim}{p}'_1, \ldots, \underset{\sim}{p}'_i$ and for $\underset{\sim}{\alpha}'_{i+1}$ this is the first time $\underset{\sim}{p}'_{i+1}$, the (i+1)th row of P is involved.

Although, $\underset{\sim}{p}'_{i+1} A_{i+1}$ can be expressed in terms of A_i, as can be seen from the next theorem, very little advantage is gained since such terms are required for each $i=0,1,\ldots,m-1$.

<u>Theorem 3.4:</u> For $i=0,1,\ldots,m-1$,

$$\underset{\sim}{p}'_{i+1} A_{i+1} = \underset{\sim}{p}'_{i+1} A_i + \underset{\sim}{\upsilon}'_i - (\underset{\sim}{p}'_{i+1} A_i \underset{\sim}{e}_{i+1}) \underset{\sim}{\upsilon}'_i . \tag{3.27}$$

<u>Proof:</u> For $i=0$,

$$\begin{aligned} \underset{\sim}{p}'_1 A_1 &= \underset{\sim}{p}'_1 [I + (\underset{\sim}{e} - \underset{\sim}{e}_1) \underset{\sim}{e}'] \ , \\ &= \underset{\sim}{p}'_1 + (\underset{\sim}{p}'_1 \underset{\sim}{e}) \underset{\sim}{e}' - (\underset{\sim}{p}'_1 \underset{\sim}{e}_1) \underset{\sim}{e}' \ , \end{aligned}$$

and the result follows, since $\underset{\sim}{p}'_1 \underset{\sim}{e} = 1$, $\underset{\sim}{\upsilon}'_o = \underset{\sim}{e}'$ and $A_o = I$.

In general, for $i=1,2,\ldots,m-1$, from (2.7) and (3.1),

$$\underset{\sim}{p}'_{i+1} A_{i+1} = \underset{\sim}{p}'_{i+1} [A_i + A_i (\underset{\sim}{e}_i - \underset{\sim}{e}_{i+1}) \underset{\sim}{\upsilon}'_i] \ .$$

Equation (3.27) follows since, from (3.19),

$$\underset{\sim}{p}'_{i+1} A_i \underset{\sim}{e}_i = \underset{\sim}{p}'_{i+1} \underset{\sim}{e} = 1 . \qquad \square$$

As a consequence of Theorem 3.3, it is suggested that (2.8) in the algorithm be replaced by (3.22).

3.4 Modification to the $\pi_m' = \pi'$ computation

At the final step of the algorithm A_m can be computed and consequently π_m' derived as $\alpha_m'/\alpha_m'e$ where $\alpha_m' = u_m'A_m$. However, A_m need not be explicitly determined since, from Theorem 3.3 and 3.4,

$$\alpha_m' = \upsilon_{m-1}' - e' + p_m'A_m,$$

where $p_m'A_m = p_m'A_{m-1} + \upsilon_{m-1}' - (p_m'A_{m-1}e_m)\upsilon_{m-1}'$.

4. Recommended procedure

As a consequence of the refinements discussed in Section 3, it is suggested that the algorithm be constructed as follows:

1. Let $A_1 = I + (e - e_1)e'$.
2. Let $\alpha_1' = p_1'A_1$.
3. For i=1,2,...,m-2, compute
   (a) $\upsilon_i' = \alpha_i'/\alpha_i'e_{i+1}$,
   (b) $B_i = A_i(e_i - e_{i+1})\upsilon_i'$,
   (c) $A_{i+1} = A_i + B_i$,
   (d) $\alpha_{i+1}' = \upsilon_i' - e' + p_{i+1}'A_{i+1}$.
4. Let $\upsilon_{m-1}' = \alpha_{m-1}'/\alpha_{m-1}'e_m$.
5. Let $\alpha_m' = 2\upsilon_{m-1}' - e' + p_m'A_{m-1} - (p_m'A_{m-1}e_m)\upsilon_{m-1}'$.
6. Let $\pi' = \pi_m' = \alpha_m'/\alpha_m'e$.

The order of the number of arithmetic operations (multiplication and division) required to determine $\underset{\sim}{\pi}'$ can be estimated as follows. The computation of the B_i and the $\underset{\sim}{p}_i'A_i$ have a dominant effect on the number of operations required. Since $A_i(\underset{\sim}{e}_i - \underset{\sim}{e}_{i+1})$ is effectively the difference of two columns of A_i, only mi operations are required to determine B_i, taking into consideration that B_i has only (i+1) non-zero rows, and, as a consequence of (3.21), that the elements of one row can be found from the other rows using the fact that each column sums to zero. On the other hand, for a general transition matrix, $\underset{\sim}{p}_i'A_i$ will require m^2 operations, although this can be reduced to m(m-1) since the ith element of this row vector, $\underset{\sim}{p}_i'A_i\underset{\sim}{e}_i = \underset{\sim}{p}_i'\underset{\sim}{e} = 1$, (by using (3.19)). Since the other calculations required are relatively insignificant in comparison, the total number of operations is of the order of

$$\sum_{i=1}^{m-1} mi + \sum_{i=1}^{m} m(m-1) = 3m^2(m-1)/2, \text{ i.e. of order } 3m^3/2.$$

To solve for the stationary distribution directly using Gaussian elimination requires of the order of $4m^3/3$, while to solve directly using a matrix inversion routine requires of the order of $2m^3$ operations, (see Isaacson and Keller (1966)).

The procedure is, in effect, finding the stationary distribution of m different irreducible Markov chains and consequently the routine that has been developed offers much more information than other techniques currently available.

Although it has been suggested that the algorithm proceed row by row, there is no necessity to adhere to a strict sequential ordering of the rows. The procedure as outlined by (2.7), (2.8) and (2.9) can easily be adapted to such changes by altering the $\underset{\sim}{t}_i$ and $\underset{\sim}{u}_i'$. A consequence of this is that the effect of changing selected transition probabilities upon the stationary distribution can easily be determined. (See also Hunter (1986)).

The procedure also offers the opportunity to utilise the structure of special transition matrices. For example, if the transition matrix of the chain is banded with $p_{ij} = 0$ for $j<i-g$ and $j>i+h$, which occurs in some queueing models, the calculation of $\underset{\sim}{p}_i A_i$ will require at most $(g+h)m$ operations and the algorithm will require on the order of only $m^3/2 + (g+h)m^2$ operations.

5. Structural results

In Section 3.2 expressions for the first few terms of the $\{A_i\}$ sequence were derived. By using those terms and working through the first few steps of the algorithm it can be shown, that, following simplification, for i=1,2,3,

$$\underset{\sim}{\alpha}_i' = (\mu_{io}\underset{\sim}{e}' + \mu_{i1}\underset{\sim}{p}_1' + \dots + \mu_{ii}\underset{\sim}{p}_i')/\mu_{ii} \, , \tag{5.1}$$

$$\underset{\sim}{\nu}_i' = (\mu_{io}\underset{\sim}{e}' + \mu_{i1}\underset{\sim}{p}_1' + \dots + \mu_{ii}\underset{\sim}{p}_i')/(\mu_{io} + \mu_{i1}p_{1,i+1} + \dots + \mu_{ii}p_{i,i+1}) \, , \tag{5.2}$$

$$\underset{\sim}{\pi}_i' = (\mu_{io}\underset{\sim}{e}' + \mu_{i1}\underset{\sim}{p}_1' + \dots + \mu_{ii}\underset{\sim}{p}_i')/(m\mu_{io} + \mu_{i1} + \dots + \mu_{ii}) \, , \tag{5.3}$$

where $\mu_{10} = 1 - p_{11}$,

$\mu_{11} = 1$,

$\mu_{20} = (1 - p_{11})(1 - p_{22}) - p_{12}p_{21}$,

$\mu_{21} = 1 + p_{21} - p_{22}$,

$\mu_{22} = 1 - p_{11} + p_{12}$.

$$\begin{aligned}\mu_{30} = {} & (1 - p_{11})(1 - p_{22})(1 - p_{33}) - p_{12}p_{23}p_{31} - p_{13}p_{21}p_{32} \\ & - p_{13}(1 - p_{22})p_{31} - (1 - p_{11})p_{23}p_{32} + p_{12}p_{21}(1 - p_{33}),\end{aligned}$$

$\mu_{31} = p_{21}(1 - p_{33} + p_{32}) + (1 - p_{22})(1 - p_{33} + p_{31}) + p_{23}(p_{31} - p_{32})$,

$\mu_{32} = p_{31}(p_{12} - p_{13}) + p_{32}(1 - p_{11} + p_{13}) + (1 - p_{33})(1 - p_{11} + p_{12})$,

$\mu_{33} = (1 - p_{11})(1 - p_{22} + p_{23}) + p_{12}(p_{23} - p_{21}) + p_{13}(1 - p_{22} + p_{21})$.

The general structure exhibited by (5.1), and hence also by (5.2) and (5.3), holds for all $i=1,2,\ldots,m$. [A proof by induction shows that if (5.1) and (5.2) hold for $i=1,\ldots,n$ then, since

$$\begin{aligned} \underset{\sim}{p}'_{n+1}A_{n+1} &= \underset{\sim}{p}'_{n+1}[I + B_o + \cdots + B_n], \\ &= \underset{\sim}{p}'_{n+1}[I + \underset{\sim}{b}_o\underset{\sim}{\upsilon}'_o + \cdots + \underset{\sim}{b}_n\underset{\sim}{\upsilon}'_n], \\ &= \underset{\sim}{p}'_{n+1} + (\underset{\sim}{p}'_{n+1}\underset{\sim}{b}_o)\underset{\sim}{\upsilon}'_o + \cdots + (\underset{\sim}{p}'_{n+1}\underset{\sim}{b}_n)\underset{\sim}{\upsilon}'_n \ . \end{aligned}$$

using (3.22) with $i=n$, $\underset{\sim}{\alpha}'_{n+1}$ is a linear combination of $\underset{\sim}{\upsilon}'_o,\ldots,\underset{\sim}{\upsilon}'_n$, $\underset{\sim}{p}'_{n+1}$, i.e. of $\underset{\sim}{e}',\underset{\sim}{p}'_1,\ldots,\underset{\sim}{p}'_{n+1}$. Furthermore, the coefficient of $\underset{\sim}{p}'_{n+1}$ is unity whereby establishing the general structure of $\underset{\sim}{\alpha}'_{n+1}$.]

Note also that, from (5.1) and (3.25), $\underset{\sim}{\alpha}'_i\underset{\sim}{e}_i = 1$, for $i=1,\ldots,m$ and thus

$$\mu_{io} + \mu_{i1}p_{1i} + \cdots + \mu_{ii}p_{ii} = \mu_{ii} \ . \tag{5.4}$$

Further, for $i=1,2$ it can be shown, by direct verification, that

$$\mu_{io} + \mu_{i1}p_{1,i+1} + \cdots + \mu_{ii}p_{i,i+1} = \mu_{i+1,i+1} \ , \tag{5.5}$$

which implies that

$$\upsilon_{ii} = \underset{\sim}{\upsilon}'_i\underset{\sim}{e}_i = \mu_{ii}/\mu_{i+1,i+1} \ , \tag{5.6}$$

and

$$\alpha_{i,i+1} = \underset{\sim}{\alpha}'_i\underset{\sim}{e}_{i+1} = \mu_{i+1,i+1}/\mu_{ii} \ . \tag{5.7}$$

results that it has not been possible to establish in general.

Let $(I-P)_i$ be the leading ith order principal submatrix of I-P formed by deleting all but the first i rows and columns then, for $i=1,2,3$,

$$\mu_{io} = \det(I-P)_i . \tag{5.8}$$

For the special case when m=3, with the notation used earlier in this section, it can be verified that

$$\underset{\sim}{\alpha}_i' = (\mu_{i1}, \mu_{i2}, \mu_{i3})/\mu_{ii}, \qquad (i=1,2,3), \tag{5.9}$$

$$\underset{\sim}{\nu}_i' = (\mu_{i1}, \mu_{i2}, \mu_{i3})/\mu_{i+1,i+1}, \qquad (i=1,2), \tag{5.10}$$

$$\underset{\sim}{\pi}_i' = (\mu_{i1}, \mu_{i2}, \mu_{i3})/(\mu_{i1} + \mu_{i2} + \mu_{i3}), \qquad (i=1,2,3), \tag{5.11}$$

where

$$\mu_{12} = 1 - p_{11} + p_{12} = \mu_{22},$$

$$\mu_{13} = 1 - p_{11} + p_{13},$$

$$\mu_{23} = \mu_{33} = 3\mu_{30}.$$

Observe that $\underset{\sim}{\pi}_1'$, $\underset{\sim}{\pi}_2'$ and $\underset{\sim}{\pi}_3'$ give, respectively, the stationary probability vectors of the Markov chains whose transition matrices are

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 1/3 & 1/3 & 1/3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}.$$

In examining (5.11), with i=3, it can be shown that $\mu_{3j} = 3D_j$, (j=1,2,3), where D_j is the determinant formed by striking out the jth row and jth column of I-P. This leads to an expression for the stationary probability vector of a general irreducible, three state, Markov chain as

$$\underset{\sim}{\pi}' = (D_1, D_2, D_3)/\sum_{j=1}^{3} D_j. \tag{5.12}$$

The natural extension of (5.12) for a general finite irreducible Markov chain is also true, such a result being attributed to Mihoc by Fréchet (1950) and rediscovered by Singer (1964).

Although the full details of a proof of the generalization of (5.12) using the techniques of this paper have not been worked out, it is conjectured that for an m-state chain $\mu_{mj} = mD_j$, (a result that holds for m=2,3), so that the procedures proposed in this paper appear to lead to an effective algorithmic construction of Mihoc's technique.

6. Final comments

The initial choice of P_o as $\underset{\sim}{e}\underset{\sim}{e}'/m$ ensures that it is possible to start with an irreducible Markov chain whose stationary distribution is easily found without having to compute a matrix inverse or to solve a general set of linear equations. The fact that every element of P_o is specified leads to a sequence of matrices $A_1, A_2, \ldots$ that are "dense". Is it possible to start with a different Markov chain, say one that is relatively sparse, whose stationary distribution is well known and such that, for the early recursions, the equivalent sequence $A_1, A_2, \ldots$ retains such a sparsity property?

The periodic Markov chain with entries $p^{(o)}_{i\,i+1}=1$, $(i=1,2,\ldots,m-1)$, and $p^{(o)}_{m1}=1$ is a potential candidate for P_o, whose stationary probability vector is also $\underset{\sim}{\pi}_o' = \underset{\sim}{e}'/m$. Even if $\underset{\sim}{t}_o$ and $\underset{\sim}{u}_o$ can be specified so that $A_o = [I - P_o + \underset{\sim}{t}_o\underset{\sim}{u}_o']^{-1}$ has a simple structure much care would be required in carrying out any sequential row modification with this P_o. For example if, for the specified P transition matrix, $p_{12} = 0$ then state 2 is never reached in the Markov chain with transition matrix P_1 violating the required irreducibility property of P_1.

The major advantage in choosing $P_o = \underset{\sim}{e}\underset{\sim}{e}'/m$ is that the irreducibility of each P_i transition matrix is guaranteed at each step of the procedure.

The next stage to be taken with the procedures proposed in this paper is to carry out some numerical studies and to compare the computational speed and accuracy with some of the other procedures surveyed in the initial section.

REFERENCES

[1] M. Fréchet, Recherches Théoriques Modernes sur le Calcul des Probabilités, Vol. 2, 2nd ed., Gauthier-Villars, Paris, 1950.

[2] G.H. Golub and C.F. Van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, MD, 1983.

[3] W.K. Grassmann, M.I. Taksar, and D.P. Heyman, Regenerative analysis and steady state distributions for Markov chains, Opns. Res. 33: 1107-1116 (1985).

[4] W.J. Harrod and R.J. Plemmons, Comparison of some direct methods for computing stationary distributions of Markov chains, SIAM J. Sci. Statist. Comput. 5: 453-469 (1984).

[5] J.J. Hunter, Generalized inverses and their application to applied probability problems, Linear Algebra Appl. 45: 157-198 (1982).

[6] J.J. Hunter, Stationary distributions of perturbed Markov chains, Linear Algebra Appl. 82: 201-214 (1986).

[7] E. Isaacson and H.B. Keller, Analysis of Numerical Methods, John Wiley and Sons, New York, 1966.

[8] J.R. Koury, D.F. McAllister and W.J. Stewart, Iterative methods for computing stationary distributions of nearly completely decomposable Markov chains, SIAM J. Alg. Disc. Meth. 5: 164-186 (1984).

[9] C.D. Meyer, Jr., Uncoupling Markov chains and the Simon-Ando theory of nearly reducible systems, Center for Research in Scientific Computation, North Carolina State Univ., Tech. Report 10018701 (1987).

[10] C.C. Paige, G.P.H. Styan, and P.G. Wachter, Computation of the stationary distribution of a Markov chain, J. Statist. Comput. Simulation 4: 173-186 (1975).

[11] T.J. Sheskin, A Markov chain partitioning algorithm for computing steady state probabilities, Opns. Res. 33: 228-235 (1985).

[12] A. Singer, The steady state probabilities of a Markov hain as a function of the transition probabilities, Opns. Res. 12: 498-499 (1964).

185. R.F. Serfozo, Point processes, May 87. Operations Research Handbook on Stochastic Processes, to appear.

186. Z.D. Bai, W.Q. Liang and W. Vervaat, Strong representation of weak convergence, June 87.

187. O. Kallenberg, Decoupling identities and predictable transformations in exchangeability, June, 87.

188. O. Kallenberg, An elementary approach to the Daniell-Kolmogorov theorem and some related results, June 87. Math. Nachr., to appear.

189. G. Samorodnitsky, Extrema of skewed stable processes, June 87.

190. D. Nualart, M. Sanz and M. Zakai, On the relations between increasing functions associated with two-parameter continuous martingales, June 87.

191. F. Avram and M. Taqqu, Weak convergence of sums of moving averages in the α-stable domain of attraction, June 87.

192. M.R. Leadbetter, Harald Cramér (1893-1985), July 87. ISI Review, to appear.

193. R. LePage, Predicting transforms of stable noise, July 87.

194. R. LePage and B.M. Schreiber, Strategies based on maximizing expected log, July 87.

195. J. Rosinski, Series representations of infinitely divisible random vectors and a generalized shot noise in Banach spaces, July 87.

196. J. Szulga, On hypercontractivity of α-stable random variables, $0<\alpha<2$, July 87.

197. I. Kuznezova-Sholpo and S.T. Rachev, Explicit solutions of moment problems I, July 87.

198. T. Hsing, On the extreme order statistics for a stationary sequence, July 87.

199. T. Hsing, Characterization of certain point processes, Aug. 87. Stochastic Proc. Appl. 26, 1987, 297-316.

200. J.P. Nolan, Continuity of symmetric stable processes, Aug. 87.

201. M. Marques and S. Cambanis, Admissible and singular translates of stable processes, Aug. 87.

202. O. Kallenberg, One-dimensional uniqueness and convergence results for exchangeable processes, Aug. 87.

203. R.J. Adler, S. Cambanis and G. Samorodnitsky, On stable Markov processes, Sept. 87.

204. G. Kallianpur and V. Perez-Abreu, Stochastic evolution equations driven by nuclear space valued martingales, Sept. 87.

205. R.L. Smith, Approximations in extreme value theory, Sept. 87.

206. E. Willekens, Estimation of convolution tails, Sept. 87.

207. J. Rosinski, On path properties of certain infinitely divisible processes, Sept. 87.

208. A.H. Korezlioglu, Computation of filters by sampling and quantization, Sept. 87.

209. J. Bather, Stopping rules and observed significance levels, Sept. 87.

210. S.T. Rachev and J.E. Yukich, Convolution metrics and rates of convergence in the central limit theorem, Sept. 87.

211. M. Fujisaki, Normed Bellman equation with degenerate diffusion coefficients and its applications to differential equations, Oct. 87.

212. G. Simons, Y.C. Yao and X. Wu, Sequential tests for the drift of a Wiener process with a smooth prior, and the heat equation, Oct. 87.

213. R.L. Smith, Extreme value theory for dependent sequences via the Stein-Chen method of Poisson approximation, Oct. 87.

214. C. Houdré, A note on vector bimeasures, Nov. 87.

215. M.R. Leadbetter, On the exceedance random measures for stationary processes, Nov. 87.

216. M. Marques, A study on Lebesgue decomposition of measures induced by stable processes, Nov. 87.

217. M.T. Alpuim, High level exceedances in stationary sequences with extremal index, Dec. 87.

218. R.F. Serfozo, Poisson functionals of Markov processes and queueing networks, Dec. 87.

219. J. Bather, Stopping rules and ordered families of distributions, Dec. 87.

220. S. Cambanis and M. Maejima, Two classes of self-similar stable processes with stationary increments, Jan. 88.

221. H.P. Hucke, G. Kallianpur and R.L. Karandikar, Smoothness properties of the conditional expectation in finitely additive white noise filtering, Jan. 88.

222. I. Mitoma, Weak solution of the Langevin equation on a generalized functional space, Feb. 88.

223. L. de Haan, S.I. Resnick, H. Rootzén and C. de Vries, Extremal behaviour of solutions to a stochastic difference equation with applications to arch-processes, Feb. 88.

224. O. Kallenberg and J. Szulga, Multiple integration with respect to Poisson and Lévy processes, Feb. 88.

225. D.A. Dawson and L.G. Gorostiza, Generalized solutions of a class of nuclear space valued stochastic evolution equations, Feb. 88.

226. G. Samorodnitsky and J. Szulga, An asymptotic evaluation of the tail of a multiple symmetric α-stable integral, Feb. 88.

227. J.J. Hunter, The computation of stationary distributions of Markov chains through perturbations, Mar. 88.

END
DATE
FILMED
DTIC
6-88